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THEORETICAL ANALYSIS OF FORCED LAMINAR CONVECTION HEAT TRANSFER IN THE ENTRANCE REGION OF AN ELLIPTIC DUCT

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INTRODUCTION

IN THIS communication forced laminar convection to a fluid of vanishing viscosity in the thermal entrance region of an elliptic duct is examined theoretically. The elliptic geometry not only has important technical applications, but is a general shape which has the circle as a particular case. It affords a convenient means of studying the effect on heat transfer of systematic departure from the circular shape. From the point of view of design, the elliptic cross section has a large perimeter to enclosed area ratio and consequently a small hydraulic diameter. This property can be used to advantage, as, for example, in compact heat exchangers or small highly rated cooled machine components.

In practice, various thermal boundary conditions and flow regimes can exist. In their studies, Dunwoody [1], Schenk and Han [2], and Rao *et al.* [3], all considered the case of fully established laminar flow. Turbulent flow has recently been studied by Cain *et al.* [4], who also detected the possible co-existence of turbulent and laminar flow conditions in their experimental work. The hypothetical slug flow model is not without interest here as it serves as a useful approximation to some of the situations in practice and a starting point in theoretical analysis for more realistic flow conditions.

THEORETICAL ANALYSIS

Using orthogonal curvilinear co-ordinates ξ_1, ξ_2 and z shown in Fig. 1, the constant property energy equation

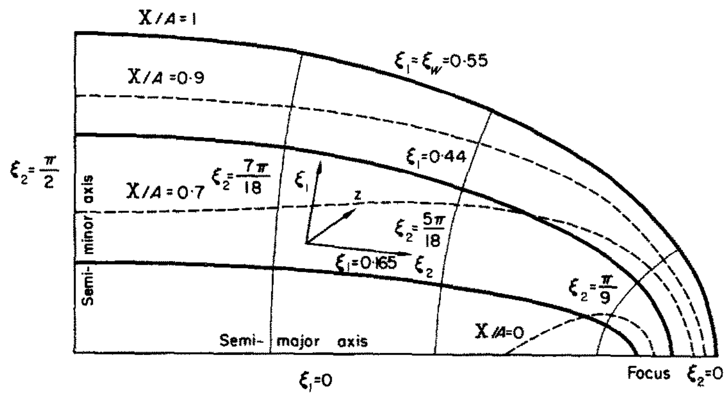

FIG. 1. Elliptic cylindrical co-ordinate system and loci of the co-ordinate variable, X (aspect ratio $= \frac{1}{2}$).

without heat generation and assuming negligible axial conduction is written:

$$\frac{u}{\alpha}\frac{\partial T}{\partial z} = \frac{1}{l_1 l_2}\left[\frac{\partial}{\partial \xi_1}\left(\frac{l_2}{l_1}\frac{\partial T}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{l_1}{l_2}\frac{\partial T}{\partial \xi_2}\right)\right] \quad (1)$$

in the usual notation.

For the particular geometry considered here, viz. the ellipse, and considering the case of slug flow, equation (1) becomes:

$$f^2(\cosh^2 \xi_1 - \cos^2 \xi_2)\bar{u}\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial \xi_1^2} + \frac{\partial^2 T}{\partial \xi_2^2}\right) \quad (2)$$

where T is the dimensionless temperature, $= \dfrac{t - t_{\text{wall}}}{t_{\text{inlet}} - t_{\text{wall}}}$

and f is the focal distance, $\bar{u}$ is the slug flow velocity, and α is the thermal diffusivity of fluid.

where $\quad \eta = (\xi_1 - \xi_w)$

$A = f^2(\cosh^2 \xi_w - \cos^2 \xi_2)$

and $\quad B = f^2 \sinh 2\xi_w$.

Equation (3) is to be solved for the dimensionless temperature, T, and the solution can be effected by the familiar separation of variables technique. The resulting integration constants are evaluated using the following boundary conditions:

(iii) $T = 0,\ X = A,\ 0 \leqslant Z$
(iv) $T = 1,\ Z = 0$, all X
where $X = A + B\eta$.
The final result is:

$$T = \sum_{n=1}^{\infty} C(n) \,.\, X^{\frac{1}{2}} \,.\, J_{-\frac{1}{3}}\left[p(n)\left(\frac{X}{A}\right)^{\frac{3}{2}}\right] \,.\, \exp\left(\frac{-9B^2 p(n) L D_e^2}{4A^3 Pr \,.\, Re}\right) \quad (4)$$

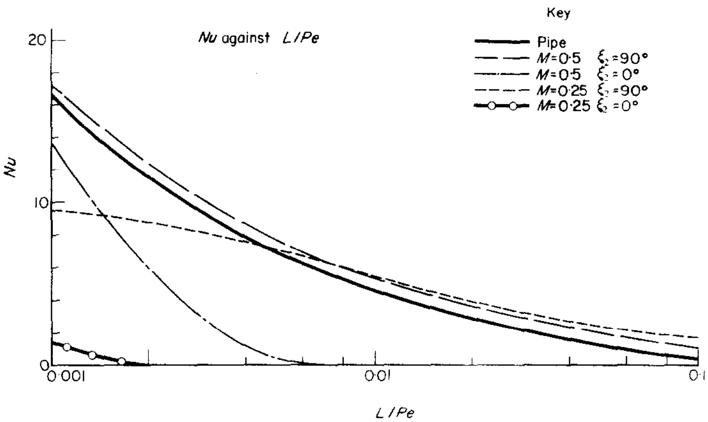

FIG. 2. Maximum and minimum heat transfer coefficients, Nu, vs the reciprocal Graetz number for various values of the aspect ratio, M.

Further simplification of the energy equation in the immediate vicinity of the entry of the duct is effected with the aid of the following:

(i) $\dfrac{\partial^2 T}{\partial \xi_2^2}$ is small compared with $\dfrac{\partial^2 T}{\partial \xi_1^2}$ and

(ii) the term $(\cosh^2 \xi_1 - \cos^2 \xi_2)$ is approximated by $(\cosh^2 \xi_w + \eta \sinh 2\xi_w - \cos^2 \xi_2)$ by the use of the first non-vanishing term of a Taylor expansion at $\xi_1 = \xi_w$ (the wall value), following [3].

Accordingly, equation (2) becomes:

$$\frac{\bar{u}}{\alpha}(A + B\eta)\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial \eta^2} \quad (3)$$

Defining Nusselt number as:

$$Nu = \left(\frac{hD_e}{k}\right) = \frac{-D_e}{l_{1w}}\left[\frac{\partial T}{\partial \xi_1}\right]_{\text{wall}} \quad (5)$$

and using the differential of T from equation (4),

$$Nu = \frac{f \sinh 2\xi_w}{\sqrt{\sinh^2 \xi_w + \sin^2 \xi_2}} \,.\, D_e \sum_{n=1}^{\infty} \frac{3}{A} \times \exp\left[\frac{-9B^2 p(n)^2 L D_e^2}{4A^3 Pe}\right] \quad (6)$$

where the $p(n)$'s are the ordered zeros of $J_{-\frac{1}{3}} = 0$, D_e is the

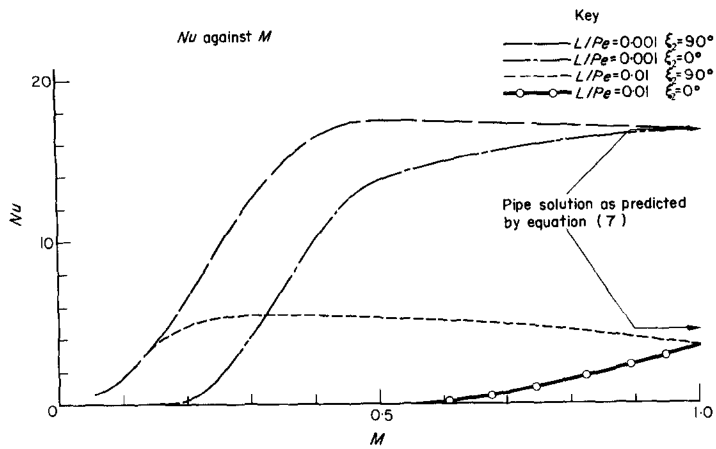

FIG. 3. Maximum and minimum heat transfer coefficients, Nu, vs the aspect ratio for various values of the reciprocal Graetz number.

equivalent diameter, L is the dimensionless distance from entry, and Pe is the Péclét number.

The corresponding expression for flow in a circular pipe using the conventional cylindrical co-ordinate system in the analysis is:

$$Nu = 4 \sum_{n=1}^{\infty} \exp\left[\frac{-4M(n)^2 L}{Pe}\right] \tag{7}$$

where the $M(n)$'s are the ordered zeros of $J_0 = 0$.

The analysis may now be applied to flow through any aspect ratio of ellipse including the limiting case, i.e. the circle. Equation (7) may be used for comparison purposes. The other limiting case of the ellipse is sometimes taken to be infinite parallel planes, but in the above analysis this does not apply.

RESULTS AND DISCUSSION

The object of the present investigation is to determine the axial and peripheral variation of local heat transfer in a slug flow in an elliptic duct of various flow conditions and geometries. With regard to peripheral variation, the values of the Nusselt number at $\xi_2 = 0$ and $\xi_2 = \pi/2$, corresponding to the ends of the major and minor axes of the ellipse respectively, are of particular interest because they represent the minimum and maximum values.

In Fig. 2, the orthodox plot of Nu vs the reciprocal of the that all the data are asymptotic to zero as L approaches (M) equal to $\frac{1}{4}$ and $\frac{1}{2}$, for the two cardinal positions on the perimeter. The limiting case of the circle (i.e. the ellipse of aspect ratio equal to unity) has been included for comparison. Recalling the present definition of Nu, it is to be observed that all the data are asymptotic to zero as L approaches infinity where the fluid temperature approaches the wall temperature and the heat transfer becomes zero. As expected, the coefficient at $\xi_2 = \pi/2$ is larger than that at $\xi_2 = 0$, the difference decreasing at large values of the reciprocal Graetz number. There are significant differences between the results for the two elliptic geometries at smaller values of $1/Gz$, and at very small values of the aspect ratio a very small value of heat transfer coefficient is anticipated, particularly at the ends of the major axis.

Figure 3 shows the local Nusselt number vs the aspect ratio for the two wall positions and for two particular values of the reciprocal Graetz number. The compatibility of the ellipse analysis (using $M = 1$) and the pipe solution (equation (7)) is seen to be good, and, as one might expect, to improve with decreasing values of (L/Pe). The curves indicate again the marked reduction in heat transfer as the ellipse becomes narrower. Extrapolation of the results to zero at $M = 0$ must not be considered to give the results for the case of infinite parallel planes. A separate analysis is required for this limiting geometry.

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